Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Combinatorics/Ordinary differential equations

Majoration of the dimension of the space of concatenated solutions to a specific pantograph equation

Majoration de la dimension de l'espace des solutions concaténées d'un cas particulier de l'équation du pantographe

Jean-François Bertazzon

Lycée Notre-Dame-de-Sion, 231, rue Paradis, 13006 Marseille, France

A R T I C L E   I N F O

A B S T R A C T

For each $\lambda \in \mathbb{N}^*$, we consider the integral equation:

$$\int\limits_{\lambda y}^{\lambda x} f(t)\, \mathrm{d}t = f(x) - f(y) \text{ for every } (x, y) \in \mathbb{R}_+^2,$$

where f is the concatenation of two continuous functions $f_a, f_b : [0, \lambda] \to \mathbb{R}$ along a word $u = u_0 u_1 \cdots \in \{a, b\}^{\mathbb{N}}$ such that $u = \sigma(u)$, where σ is a λ-uniform substitution satisfying some combinatorial conditions.

There exists some non-trivial solutions ([1]). We show in this work that the dimension of the set of solutions is at most two.

© 2018 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Nous considérons les équations intégrales de la forme suivante pour $\lambda \in \mathbb{N}^*$:

$$\int\limits_{\lambda y}^{\lambda x} f(t)\, \mathrm{d}t = f(x) - f(y) \text{ for every } (x, y) \in \mathbb{R}_+^2,$$

où f est la concaténation de deux fonctions continues $f_a, f_b : [0, \lambda] \to \mathbb{R}$ le long d'un mot infini $u = u_0 u_1 \cdots \in \{a, b\}^{\mathbb{N}}$ tel que $u = \sigma(u)$, où σ est une substitution λ-uniforme vérifiant certaines propriétés combinatoires.

Il existe des solutions non triviales à ces équations ([1]). Nous montrons dans ce travail que l'espace des solutions est de dimension au plus 2.

© 2018 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

For each positive integer $\lambda \geq 2$ and each integer $\delta \in \mathbb{Z}^*$, we consider the integral equation:

$$\int_{\lambda y}^{\lambda x} f(t) \, dt = \delta \big(f(x) - f(y) \big) \text{ for every } (x, y) \in (\mathbb{R}_+)^2. \tag{$E_{\lambda, \delta}$}$$

This equation is a particular case of the *pantograph* equation:

$$f'(x) = af(\tau x) + bf(x) \quad \text{with } (a, b) \in \mathbb{R}^2 \text{ and } \tau \in \mathbb{R}_+ \text{ for } x \geq 0.$$

We refer to [2], [4], [5] and [6] for more details on the pantograph equation.

We prove in [1] that we can extend each continuous function f defined on $[1, \lambda]$ such that $f^{(n)}(1) = f^{(n)}(\lambda) = 0$ for every non-negative integer n, into a continuous solution to $(E_{\lambda, \delta})$. Therefore, the set of continuous solutions to $(E_{\lambda, \delta})$ is an infinite-dimensional vector space.

Moreover, we prove in [1] that the non-identically zero solutions are not periodic. It seems natural to look for the simplest solutions to $(E_{\lambda, \delta})$. The periodic functions are the repetition of the same motif. We study the functions that are the repetition (not periodically) of two functions. This leads us to the following notion of *concatenation* of two functions along a word.

Definition 1.1. Let $\lambda \geq 2$ be a positive integer and $f_a, f_b : [0, \lambda] \to \mathbb{R}$ be two functions. For each finite word $u = u_0 \cdots u_{n-1} \in \{a, b\}^n$ of length n, we define a function $f_u : [0, n\lambda] \to \mathbb{R}$ called *the concatenation of* f_a *and* f_b *along* u by:

$$f_u(x + \lambda k) := f_{u_k}(x) \text{ for } x \in [0, \lambda] \text{ and } k \in \{0, \ldots, n-1\}.$$

We extend this definition to infinite words.

Our main result is the following theorem. We recall in Section 2 some notions of combinatorics on words requisite to fully understand this result.

Theorem 1.2. *We consider a λ-uniform substitution σ, satisfying some combinatorial conditions (Relations (1) and (3)) and $u = u_0 u_1 \cdots \in \{a, b\}^{\mathbb{N}}$ an infinite word such that $u = \sigma(u)$.*

We consider the integral equation:

$$\int_{\lambda y}^{\lambda x} f(t) \, dt = f(x) - f(y) \text{ for every } (x, y) \in (\mathbb{R}_+)^2. \tag{E_λ}$$

We denote by $\mathcal{S}_\lambda$ the set of solutions f to (E_λ) that are the concatenation of two continuous functions $f_a, f_b : [0, \lambda] \to \mathbb{R}$ along the word u. Then $\mathcal{S}_\lambda$ is a vector space of dimension at most 2.

We prove in [1] that $\mathcal{S}_\lambda$ is of dimension at least 1. To construct a non-trivial solution, we renormalized some iterated Birkhoff sums. The technique used to prove Theorem 1.2 (in Section 5) is very different. It is based on the relation between the values taken by the functions and their moments. This brings us back to the historical first non-trivial solution associated with the Prouhet–Thue–Morse substitution ($a \to ab$ and $b \to ba$) constructed by Fabius ([3]) as a cumulative distribution function.

We do not have examples of substitutions for which the dimension of $\mathcal{S}_\lambda$ is two.

We will use the two following basic results (see [1]).

Remark 1. Let f be as in Theorem 1.2, then for every finite word v of length n:

$$\int_{\lambda y}^{\lambda x} f_{\sigma(v)}(t) \, dt = f_v(x) - f_v(y) \text{ for every } (x, y) \in [0, n\lambda]^2.$$

Remark 2. We have $f_a(0) = f_a(\lambda) = f_b(0) = f_b(\lambda)$.

2. Some notions about combinatorics on words

We consider the *alphabet* $\{a, b\}$ consisting of two *letters* a and b. We denote by $\{a, b\}^*$ the set of *finite words*. Endowed with the concatenation, it is a free monoid and an endomorphism is called a *substitution*. If u is a finite word, we denote by $|u|$ its *length* and $|u|_\alpha$ the number of occurrences of the letter α for $\alpha \in \{a, b\}$.

A substitution σ is said to be λ-*uniform* if $\lambda := |\sigma(a)| = |\sigma(b)|$. We only consider λ-uniform substitutions σ such that:

$$\lambda_a := |\sigma(a)|_a = |\sigma(b)|_a \quad \text{and} \quad \lambda_b := |\sigma(a)|_b = |\sigma(b)|_b. \tag{1}$$

We have, of course, $\lambda_a + \lambda_b = \lambda$. The next notion takes care of the order of apparition of the letters in $\sigma(a)$ and $\sigma(b)$. If $u = u_0 \cdots u_{n-1}$ is a finite word with $n > 1$, we define the set of strict prefixes by $\mathrm{pref}(u) := \{u_0 \cdots u_k; 0 \le k < n - 1\}$ and

$$\delta^{(1)}_{a \in \sigma(a)} := \sum_{v \in \mathrm{pref}(\sigma(a))} |v|_a \text{ and } \delta^{(1)}_{a \in \sigma(b)} := \sum_{v \in \mathrm{pref}(\sigma(b))} |v|_b. \tag{2}$$

We assume that:

$$\delta := \delta^{(1)}_{a \in \sigma(a)} - \delta^{(1)}_{a \in \sigma(b)} = 1. \tag{3}$$

We fix for the rest of this work such a substitution σ.

Lemma 2.1. *Let $\alpha \in \{a, b\}$ be a letter and χ_α be the function defined for a word (finite or infinite) $v = v_0 v_1 \cdots$ by:*

$$\chi_\alpha(v) = 1 \text{ if } v_0 = \alpha \quad \text{and} \quad \chi_\alpha(v) = 0 \text{ otherwise}.$$

If $u = u_0 u_1 \cdots$ is a word (finite or infinite), we define the (left) shift by $S(u) = u_1 u_2 \cdots$. The terms $\delta^{(1)}_{\alpha \in \sigma(a)}$ are double Birkhoff sums:

$$\delta^{(1)}_{\alpha \in \sigma(a)} = \sum_{k=0}^{\lambda-1} (\lambda - k - 1) \chi_\alpha\big(S^k \circ \sigma(a)\big) = \sum_{k=1}^{\lambda-1} \sum_{i=0}^{k-1} \chi_\alpha\big(S^i \circ \sigma(a)\big) \quad \text{for every } \alpha \in \{a, b\}. \tag{4}$$

Definition 2.2. We generalize Equation (4) to every positive integer ℓ by:

$$\delta^{(\ell)}_{\alpha \in \sigma(\beta)} := \frac{1}{\ell!} \sum_{k=0}^{\lambda-1} (\lambda - k - 1)^\ell \chi_\alpha\big(S^k \circ \sigma(\beta)\big) \quad \text{for two letters } (\alpha, \beta) \in \{a, b\}^2. \tag{5}$$

These terms are closed, but different from the iterated Birkhoff sums over $\sigma(\alpha)$ introduced in [1] if $\ell > 2$. By convention, we define $\delta^{(0)}_{a \in \sigma(a)} := \lambda_a$ and $\delta^{(0)}_{b \in \sigma(a)} := \lambda_b$. It is clear that for every positive integer ℓ and every letter $\alpha \in \{a, b\}$,

$$\delta^{(\ell)}_{a \in \sigma(\alpha)} + \delta^{(\ell)}_{b \in \sigma(\alpha)} = \frac{1}{\ell!} \sum_{k=0}^{\lambda-1} (\lambda - k - 1)^\ell. \tag{6}$$

Note that it does not depend on the substitution.

3. Definition of normalized moments

Definition 3.1. Let σ be a λ-uniform substitution satisfying (1) and (3). Let f be a solution to (E_λ), which is the concatenation of two continuous functions $f_a, f_b : [0, \lambda] \to \mathbb{R}$ along a word $u = u_0 u_1 \cdots \in \{a, b\}^\mathbb{N}$ such that $u = \sigma(u)$. We define the ℓ-*th moment* for $\ell \in \mathbb{N}$ by:

$$m^{(\ell)}_\alpha := \int_0^\lambda \big(\lambda - x\big)^\ell \cdot f_\alpha(x) \, dx \quad \text{for } \alpha \in \{a, b\}.$$

We also define the ℓ-*th normalized moment* for $\ell \in \mathbb{N}$ by:

$$\tilde{m}^{(\ell)}_\alpha := \frac{1}{\ell!} \frac{1}{\lambda^\ell} m^{(\ell)}_\alpha = \frac{1}{\ell!} \frac{1}{\lambda^\ell} \int_0^\lambda \big(\lambda - x\big)^\ell \cdot f_\alpha(x) \, dx \quad \text{for } \alpha \in \{a, b\}.$$

Lemma 3.2. *For every non-negative integer $\ell \in \mathbb{N}$ and every letter $\alpha \in \{a, b\}$:*

$$\sum_{q=0}^{\ell+1} (-1)^q \left(\delta_{a \in \sigma(\alpha)}^{(\ell+1-q)} \tilde{m}_a^{(q)} + \delta_{b \in \sigma(\alpha)}^{(\ell+1-q)} \tilde{m}_b^{(q)} \right) = -\frac{1}{\ell!} \lambda^{\ell+2} f(0) + \lambda^{\ell+1} \tilde{m}_\alpha^{(\ell)}. \tag{7}$$

Proof of Lemma 3.2. We fix a letter $\alpha \in \{a, b\}$ and a non-negative integer $\ell \in \mathbb{N}$. From Remark 1, the function $F(x) = \lambda f_\alpha \left(\frac{x}{\lambda} \right)$ is a primitive function of $f_{\sigma(\alpha)}$. We calculate the following integral, recalling that $f_{\sigma(\alpha)}(0) = f_\alpha(0) = f(0)$ (Remark 2):

$$\int_0^{\lambda^2} \left(\lambda^2 - x \right)^{\ell+1} \cdot f_{\sigma(\alpha)}(x) \, \mathrm{d}x = \left[\left(\lambda^2 - x \right)^{\ell+1} \cdot F(x) \right]_0^{\lambda^2} + (\ell+1) \int_0^{\lambda^2} \left(\lambda^2 - x \right)^\ell \cdot F(x) \, \mathrm{d}x$$

$$= -\lambda^{2(\ell+1)} \cdot \lambda f_\alpha \left(\frac{0}{\lambda} \right) + (\ell+1) \int_0^{\lambda^2} \left(\lambda^2 - x \right)^\ell \cdot \lambda f_\alpha \left(\frac{x}{\lambda} \right) \, \mathrm{d}x$$

$$= -\lambda^{2\ell+3} \cdot f(0) + (\ell+1) \cdot \lambda^2 \int_0^\lambda \left(\lambda^2 - \lambda x \right)^\ell \cdot f_\alpha(x) \, \mathrm{d}x$$

$$= -\lambda^{2\ell+3} \cdot f(0) + (\ell+1) \cdot \lambda^{\ell+2} m_\alpha^{(\ell)}. \tag{8}$$

We write $\sigma(\alpha) = v_0 \cdots v_{\lambda-1}$, and with Definition 1.1:

$$\int_0^{\lambda^2} \left(\lambda^2 - x \right)^{\ell+1} \cdot f_{\sigma(\alpha)}(x) \, \mathrm{d}x = \sum_{k=0}^{\lambda-1} \int_{k\lambda}^{(k+1)\lambda} \left(\lambda^2 - x \right)^{\ell+1} \cdot f_{\sigma(\alpha)}(x) \, \mathrm{d}x$$

$$= \sum_{k=0}^{\lambda-1} \int_0^\lambda \left(\lambda^2 - x - k\lambda \right)^{\ell+1} \cdot f_{v_k}(x) \, \mathrm{d}x.$$

It remains to express $(\lambda^2 - x - k\lambda)^{\ell+1}$ in the basis $((\lambda - x)^q; 0 \le q \le \ell+1)$. To do this, we derive q times the polynomial function $(\lambda^2 - x - k\lambda)^{\ell+1}$, and we estimate it at $x = \lambda$. We fix $k \in \{0, \ldots, \lambda-1\}$:

$$\left(\lambda^2 - x - k\lambda \right)^{\ell+1} = \sum_{q=0}^{\ell+1} a_{q,k}^{(\ell+1)} (\lambda - x)^q,$$

where $a_{q,k}^{(\ell+1)} = \frac{(\ell+1)!}{q! \, (\ell+1-q)!} (-1)^q \left(\lambda^2 - \lambda(k+1) \right)^{\ell+1-q}$ for $0 \le q \le \ell+1$. In particular,

$$a_{\ell+1,k}^{(\ell+1)} = (-1)^{\ell+1}, \; a_{\ell,k}^{(\ell+1)} = (\ell+1) \cdot (-1)^\ell \cdot \lambda \cdot \left(\lambda - k - 1 \right), \; \ldots$$

Therefore, we have:

$$\int_0^{\lambda^2} \left(\lambda^2 - x \right)^{\ell+1} \cdot f_{\sigma(\alpha)}(x) \, \mathrm{d}x = \sum_{k=0}^{\lambda-1} \sum_{q=0}^{\ell+1} a_{q,k}^{(\ell+1)} \int_0^\lambda \left(\lambda - x \right)^q \cdot f_{v_k}(x) \, \mathrm{d}x. \tag{9}$$

With Equations (8), we find:

$$\sum_{q=0}^{\ell+1} \sum_{k=0}^{\lambda-1} a_{q,k}^{(\ell+1)} m_{v_k}^{(q)} = -\lambda^{2\ell+3} f(0) + (\ell+1) \lambda^{\ell+2} m_\alpha^{(\ell)}.$$

The normalized relation is:

$$\sum_{q=0}^{\ell+1} \frac{(-1)^q}{(\ell+1-q)!} \sum_{k=0}^{\lambda-1} \left(\lambda - k - 1 \right)^{\ell+1-q} \tilde{m}_{v_k}^{(q)} = -\frac{1}{\ell!} \lambda^{\ell+2} f(0) + \frac{1}{\ell!} \lambda m_\alpha^{(\ell)}.$$

We simplify this expression with Definition 2.2 of $\delta_{\beta\in\sigma(\alpha)}^{(\ell+1-q)}$:

$$\sum_{q=0}^{\ell+1}(-1)^q\left(\delta_{a\in\sigma(\alpha)}^{(\ell+1-q)}\tilde{m}_a^{(q)}+\delta_{b\in\sigma(\alpha)}^{(\ell+1-q)}\tilde{m}_b^{(q)}\right)=-\frac{1}{\ell!}\lambda^{\ell+2}\,f(0)+\lambda^{\ell+1}\tilde{m}_\alpha^{(\ell)}.\quad\square$$

4. A technical lemma

Lemma 4.1. *Let λ be a positive real number and f be a continuous function which is solution to (E_λ). Then, for every $n\in\mathbb{N}$ and $\ell\in\mathbb{N}$,*

$$f\left(\frac{n+1}{\lambda^{\ell+1}}\right)-f\left(\frac{n}{\lambda^{\ell+1}}\right)=\sum_{k=0}^{\ell}\frac{1}{(k+1)!}\frac{1}{\lambda^{(k+1)\left(\ell-\frac{k}{2}\right)}}f\left(\frac{n}{\lambda^{\ell-k}}\right)+I_n^{(\ell+1)},\tag{10}$$

where the remainder integral $I_n^{(\ell)}$ is:

$$I_n^{(\ell)}:=\frac{1}{\lambda^{\ell(\ell+1)/2}}\frac{1}{\ell!}\int_0^\lambda(\lambda-u)^\ell f(u+\lambda n)\,\mathrm{d}u.\tag{11}$$

Remark 3. Let σ be a λ-uniform substitution satisfying (1) and (3). We suppose that f is a solution to (E_λ) that is the concatenation of two continuous functions $f_a, f_b:[0,\lambda]\to\mathbb{R}$ along a word $u=u_0u_1\cdots\in\{a,b\}^\mathbb{N}$ such that $u=\sigma(u)$.

Then, for every $n\in\mathbb{N}$ and $\ell\in\mathbb{N}$, $I_n^{(\ell)}$ depends only on u_n and ℓ. With Definition 3.1 of moments, Relation (10) can be rewritten as follows:

$$f\left(\frac{n+1}{\lambda^{\ell+1}}\right)-f\left(\frac{n}{\lambda^{\ell+1}}\right)=\sum_{k=0}^{\ell}\frac{1}{(k+1)!}\frac{1}{\lambda^{(k+1)\left(\ell-\frac{k}{2}\right)}}f\left(\frac{n}{\lambda^{\ell-k}}\right)+\frac{1}{\lambda^{\ell(\ell-1)/2}}\tilde{m}_{v_n}^{(\ell)}.\tag{12}$$

Proof of Lemma 4.1. We fix two non-negative integers n and ℓ. From Equation (E_λ):

$$f\left(\frac{n+1}{\lambda^{\ell+1}}\right)-f\left(\frac{n}{\lambda^{\ell+1}}\right)=\int_{n/\lambda^\ell}^{(n+1)/\lambda^\ell}f(t)\,\mathrm{d}t.$$

Still according to Equation (E_λ), the values of the function at $n\cdot\lambda^{-\ell}$ and $t\in\mathbb{R}_+$ satisfy:

$$\forall t\in\mathbb{R}_+,\quad f(t)=f\left(\frac{n}{\lambda^\ell}\right)+\int_{n/\lambda^{\ell-1}}^{\lambda t}f(s_1)\,\mathrm{d}s_1.$$

The two previous relations involve:

$$f\left(\frac{n+1}{\lambda^{\ell+1}}\right)-f\left(\frac{n}{\lambda^{\ell+1}}\right)=\frac{1}{\lambda^\ell}f\left(\frac{n}{\lambda^\ell}\right)+\int_{n/\lambda^\ell}^{(n+1)/\lambda^\ell}\int_{n/\lambda^{\ell-1}}^{\lambda t}f(s_1)\,\mathrm{d}s_1\,\mathrm{d}t.$$

We can iterate the process using the relation:

$$\text{for }s_1\in\mathbb{R}_+,\quad f(s_1)=f\left(\frac{n}{\lambda^{\ell-1}}\right)+\int_{n/\lambda^{\ell-2}}^{\lambda s_1}f(s_2)\,\mathrm{d}s_2.$$

The goal is to continue this process (like Taylor series) and to express $f\left(\frac{n+1}{\lambda^{\ell+1}}\right)-f\left(\frac{n}{\lambda^{\ell+1}}\right)$ as a linear combination of $\{f(n\cdot\lambda^{-k});0\le k\le\ell\}$ and an integral on $[k\lambda,(k+1)\lambda]$. To do this, we introduce the following terms:

$$V_k:=\int_{n/\lambda^\ell}^{(n+1)/\lambda^\ell}\int_{n/\lambda^{\ell-1}}^{\lambda t}\int_{n/\lambda^{\ell-2}}^{\lambda s_1}\cdots\int_{n/\lambda^{\ell-k}}^{\lambda s_{k-1}}\mathrm{d}s_k\cdots\mathrm{d}s_1\,\mathrm{d}t\quad\text{for }k\in\{0,\ldots,\ell\},\tag{13}$$

$$I := \int\limits_{n/\lambda^\ell}^{(n+1)/\lambda^\ell} \int\limits_{n/\lambda^{\ell-1}}^{\lambda t} \int\limits_{n/\lambda^{\ell-2}}^{\lambda s_1} \cdots \int\limits_{\lambda n}^{\lambda s_\ell} f(s_{\ell+1}) \, ds_{\ell+1} \cdots ds_1 \, dt. \tag{14}$$

Continuing the process described above, we have:

$$f\left(\frac{n+1}{\lambda^{\ell+1}}\right) - f\left(\frac{n}{\lambda^{\ell+1}}\right) = \sum_{k=0}^{\ell} V_k f\left(\frac{n}{\lambda^{\ell-k}}\right) + I. \tag{15}$$

It remains to calculate the values of $(V_k)_{0 \le k \le \ell}$ and I.

– We fix $k \in \{0, \dots, \ell\}$ and we calculate V_k defined in (13). We put $u_i = \lambda(\lambda^{\ell-i} s_i - n)$ for $i \in \{1, \dots, k-1\}$ and $v = \lambda(\lambda^\ell t - n)$:

$$V_k = \prod_{i=0}^{k} \frac{1}{\lambda^{\ell-i+1}} \int\limits_0^\lambda \int\limits_0^v \int\limits_0^{u_1} \cdots \int\limits_0^{u_{k-1}} du_k \cdots du_1 \, dv.$$

$$= \frac{1}{\lambda^{(k+1)(\ell+1)}} \left(\prod_{i=0}^{k} \lambda^i\right) \frac{\lambda^{k+1}}{(k+1)!} = \frac{1}{(k+1)!} \lambda^{(k+1)\left(\frac{k}{2} - \ell\right)}. \tag{16}$$

– We make the substitution $u + \lambda n = s_{\ell+1}$ in Equation (14):

$$I = \int\limits_{n/\lambda^\ell}^{(n+1)/\lambda^\ell} \int\limits_{k/\lambda^{\ell-1}}^{\lambda t} \int\limits_{n/\lambda^{\ell-2}}^{\lambda s_1} \cdots \int\limits_{\lambda n}^{\lambda s_\ell} f(u + \lambda n) \, du \, ds_\ell \cdots ds_1 \, dt.$$

If we put $u_i = \lambda(\lambda^{\ell-i} s_i - n)$ for $i \in \{1, \dots, \ell\}$ and $v = \lambda(\lambda^\ell t - n)$:

$$I = \prod_{i=0}^{\ell+1} \frac{1}{\lambda^{\ell-i+1}} \int\limits_0^\lambda \int\limits_0^v \int\limits_0^{u_1} \cdots \int\limits_0^{u_{\ell-1}} f(u + \lambda n) \, du \, du_\ell \cdots du_1 \, dv$$

$$= \prod_{i=0}^{\ell+1} \frac{1}{\lambda^i} \int\limits_0^\lambda \left(\int\limits_{u < u_\ell < \cdots < u_1 < v < \lambda} dv \, du_1 \cdots du_\ell\right) f(u + \lambda n) \, du$$

$$= \frac{1}{\lambda^{(\ell+1)(\ell+2)/2}} \int\limits_0^\lambda \frac{(\lambda - u)^{\ell+1}}{(\ell+1)!} f(u + \lambda n) \, du = I_n^{(\ell+1)}. \tag{17}$$

Lemma 4.1 is proved by combining Relations (15), (16), and (17). $\quad\square$

5. Proof of Theorem 1.2

Let σ be a λ-uniform substitution satisfying (1) and (3). We denote by $\mathcal{S}_\lambda$ the set of solutions to (E_λ) that are the concatenation of two continuous functions $[0, \lambda] \to \mathbb{R}$ along a word $u = u_0 u_1 \cdots \in \{a, b\}^{\mathbb{N}}$ such that $u = \sigma(u)$.
We prove that the map from $\mathcal{S}_\lambda$ into $\mathbb{R}^2$ defined by $f \mapsto (f(0), f(1))$ is an injective morphism.
We fix a function $f \in \mathcal{S}_\lambda$ such that $f(0) = f(1) = 0$.

(i) From Equation (E_λ):

$$m_{u_0}^{(0)} = \tilde{m}_{u_0}^{(0)} = \int\limits_0^\lambda f(t) \, dt = f(1) - f(0) = 0.$$

We calculate the following integral for every non-negative integer n:

$$\int\limits_0^{n\lambda^2} f(t) \, dt = \begin{cases} \sum\limits_{k=0}^{n-1} \sum\limits_{i=0}^{\lambda-1} \int\limits_{k^2\lambda + i\lambda}^{k^2\lambda + (i+1)\lambda} f(t) \, dt = \sum\limits_{k=0}^{n-1} \left(\lambda_a m_a^{(0)} + \lambda_b m_b^{(0)}\right) = n\left(\lambda_a m_a^{(0)} + \lambda_b m_b^{(0)}\right), \\ f(n\lambda) - f(0). \end{cases}$$

We divide this expression by n and since f is bounded:

$$\lambda_a m_a^{(0)} + \lambda_b m_b^{(0)} = \frac{1}{n} f(n\lambda) - \frac{1}{n} f(0) \underset{n \to +\infty}{\longrightarrow} 0.$$

So we have $m_a^{(0)} = m_b^{(0)} = 0$ and for every $n \in \mathbb{N}$:

$$f(n) = f(0) + \int_0^{n\lambda} f(t)\,\mathrm{d}t = f(0) + \sum_{k=0}^{n-1} \int_0^{\lambda} f(t)\,\mathrm{d}t = f(0) + \sum_{k=0}^{n-1} m_{u_k}^{(0)}$$

$$= f(0) + |u_0 \cdots u_{n-1}|_a \cdot m_a^{(0)} + |u_0 \cdots u_{n-1}|_b \cdot m_b^{(0)} = 0.$$

(ii) *We show by induction on $\ell \geq 1$ that $m_\alpha^{(i)} = 0$ for every $\alpha \in \{a, b\}$ and every $0 \leq i \leq \ell$.*
We suppose that $m_\alpha^{(i)} = 0$ for $\alpha \in \{a, b\}$ and $0 \leq i \leq \ell$. We recall Relation (7) in Lemma 3.2 for $\alpha = a$:

$$\sum_{q=0}^{\ell+1} (-1)^q \left(\delta_{a \in \sigma(a)}^{(\ell+1-q)} \tilde{m}_a^{(q)} + \delta_{b \in \sigma(a)}^{(\ell+1-q)} \tilde{m}_b^{(q)} \right) = -\frac{1}{\ell!} \lambda^{\ell+2} f(0) + \lambda^{\ell+1} \tilde{m}_a^{(\ell)}.$$

By induction hypothesis, we have:

$$\lambda_a \tilde{m}_a^{(\ell+1)} + \lambda_b \tilde{m}_b^{(\ell+1)} = 0. \tag{18}$$

We use Relation (7) for the positive integer $\ell + 1$ with $\alpha = a$ and $\alpha = b$:

$$\begin{cases} \displaystyle\sum_{q=0}^{\ell+2} (-1)^q \left(\delta_{a \in \sigma(a)}^{(\ell+2-q)} \tilde{m}_a^{(q)} + \delta_{b \in \sigma(a)}^{(\ell+2-q)} \tilde{m}_b^{(q)} \right) = -\frac{1}{(\ell+1)!} \lambda^{\ell+3} f(0) + \lambda^{\ell+2} \tilde{m}_a^{(\ell+1)}, \\ \displaystyle\sum_{q=0}^{\ell+2} (-1)^q \left(\delta_{a \in \sigma(b)}^{(\ell+2-q)} \tilde{m}_a^{(q)} + \delta_{b \in \sigma(b)}^{(\ell+2-q)} \tilde{m}_b^{(q)} \right) = -\frac{1}{(\ell+1)!} \lambda^{\ell+3} f(0) + \lambda^{\ell+2} \tilde{m}_b^{(\ell+1)}. \end{cases}$$

$$\Longleftrightarrow \begin{cases} \displaystyle\sum_{q=\ell+1}^{\ell+2} (-1)^q \left(\delta_{a \in \sigma(a)}^{(\ell+2-q)} \tilde{m}_a^{(q)} + \delta_{b \in \sigma(a)}^{(\ell+2-q)} \tilde{m}_b^{(q)} \right) = \lambda^{\ell+2} \tilde{m}_a^{(\ell+1)}, \\ \displaystyle\sum_{q=\ell+1}^{\ell+2} (-1)^q \left(\delta_{a \in \sigma(b)}^{(\ell+2-q)} \tilde{m}_a^{(q)} + \delta_{b \in \sigma(b)}^{(\ell+2-q)} \tilde{m}_b^{(q)} \right) = \lambda^{\ell+2} \tilde{m}_b^{(\ell+1)}. \end{cases}$$

When we subtract these two relations, the coefficients of $\tilde{m}_a^{(\ell+2)}$ and $\tilde{m}_a^{(\ell+2)}$ vanish:

$$(-1)^{\ell+1} \cdot \left(\delta_{a \in \sigma(a)}^{(1)} - \delta_{a \in \sigma(b)}^{(1)} \right) \tilde{m}_a^{(\ell+1)} + (-1)^{\ell+1} \cdot \left(\delta_{b \in \sigma(a)}^{(1)} - \delta_{b \in \sigma(b)}^{(1)} \right) \tilde{m}_a^{(\ell+1)} = \lambda^{\ell+2} \left(\tilde{m}_a^{(\ell+1)} - \tilde{m}_b^{(\ell+1)} \right). \tag{19}$$

We recall that $\delta_{a \in \sigma(a)}^{(1)} - \delta_{a \in \sigma(b)}^{(1)} = 1$. Moreover, from (6),

$$\delta_{a \in \sigma(a)}^{(1)} + \delta_{b \in \sigma(a)}^{(1)} = \delta_{a \in \sigma(b)}^{(1)} + \delta_{b \in \sigma(b)}^{(1)}.$$

Then $\delta_{a \in \sigma(b)}^{(1)} - \delta_{a \in \sigma(a)}^{(1)} = -1$. We can rewrite (19):

$$(-1)^{\ell+1} \tilde{m}_a^{(\ell+1)} - (-1)^{\ell+1} \tilde{m}_a^{(\ell+1)} = \lambda^{\ell+2} \left(\tilde{m}_a^{(\ell+1)} - \tilde{m}_b^{(\ell+1)} \right),$$

so we find:

$$\tilde{m}_a^{(\ell+1)} - \tilde{m}_b^{(\ell+1)} = 0. \tag{20}$$

Combining Equations (18) and (20): $\tilde{m}_a^{(\ell+1)} = \tilde{m}_b^{(\ell+1)} = 0$.

(iii) *We show by induction on $\ell \in \mathbb{N}$ that $f(n\lambda^{-i}) = 0$ for every $n \in \mathbb{N}$ and $0 \leq i \leq \ell$.*
For every $n \in \mathbb{N}$, we recall Equation (12):

$$f\left(\frac{n+1}{\lambda^{\ell+1}} \right) - f\left(\frac{n}{\lambda^{\ell+1}} \right) = \sum_{k=0}^{\ell} \frac{1}{(k+1)!} \frac{1}{\lambda^{(k+1)\left(\ell - \frac{k}{2}\right)}} f\left(\frac{n}{\lambda^{\ell-k}} \right).$$

It is easy to verify by induction that, for every $n \in \mathbb{N}$,

$$f\left(\frac{n+1}{\lambda^{\ell+1}}\right) - f\left(\frac{n}{\lambda^{\ell+1}}\right) = 0.$$

Since $f(0) = 0$, then $f\left(\frac{n}{\lambda^{\ell+1}}\right) = 0$ for every $n \in \mathbb{N}$.

(iv) We have seen that f vanishes at λ-adic points (i.e. the points $n\lambda^{-k}$ for $n, k \in \mathbb{N}$). These points form a dense subset of $\mathbb{R}_+$ and f is a continuous function; therefore, f is the identically zero function.

References

[1] J.-F. Bertazzon, V. Delecroix, Sommes de Birkhoff itérées sur des extensions finies d'odomètres. Construction de solutions auto-similaires à des équations différentielles avec délai, Bull. Soc. Math. Fr. (2018), in press.

[2] L. Bogachev, G. Derfel, S. Molchanov, J. Ockendon, On bounded solutions of the balanced generalized pantograph equation, in: Topics in Stochastic Analysis and Nonparametric Estimation, in: The IMA Volumes in Mathematics and its Applications, vol. 145, 2008, pp. 24–49.

[3] J. Fabius, A probabilistic example of a nowhere analytic C^∞-function, Z. Wahrscheinlichkeitstheor. Verw. Geb. 5 (2) (1966) 173–174.

[4] A. Saadatmandi, M. Dehghan, Variational iteration method for solving a generalized pantograph equation, Comput. Math. Appl. 58 (11) (2002) 2190–2196.

[5] T. Yoneda, On the functional-differential equation of advanced type $f'(x) = af(2x)$ with $f(0) = 0$, J. Math. Anal. Appl. 37 (1) (2006) 320–330.

[6] T. Yoneda, On the functional-differential equation of advanced type $f'(x) = af(\lambda x)$, $\lambda > 1$, with $f(0) = 0$, J. Math. Anal. Appl. 332 (1) (2007) 487–496.