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Interaction energy between vortices of vector fields on Riemannian surfaces



Énergie d'interaction entre les tourbillons des champs de vecteurs sur une surface riemannienne

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ABSTRACT

We study a variational Ginzburg–Landau-type model depending on a small parameter $\varepsilon > 0$ for (tangent) vector fields on a 2-dimensional Riemannian surface. As $\varepsilon \rightarrow 0$, the vector fields tend to be of unit length and will have singular points of a (non-zero) index, called vortices. Our main result determines the interaction energy between these vortices as a Γ -limit (at the second order) as $\varepsilon \rightarrow 0$.

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R É S U M É

Nous étudions un modèle variationnel de type Ginzburg–Landau (dépendant d'un petit paramètre $\varepsilon > 0$) pour des champs de vecteurs (tangents) sur une surface riemannienne. Lorsque $\varepsilon \rightarrow 0$, ces champs de vecteurs auront des points singuliers d'indice non nul, appelés tourbillons. Notre résultat détermine l'énergie d'interaction entre les tourbillons en tant que Γ -limite (au second ordre) pour $\varepsilon \rightarrow 0$.

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Soit (S, g) une surface riemannienne orientée compacte, connexe, sans bord, de dimension 2 et de genre g . Nous considérons la fonctionnelle de Ginzburg–Landau E_ε (voir (1)) dépendant d'un petit paramètre $\varepsilon > 0$ pour des champs de vecteurs (tangents) $u : S \rightarrow TS$, i.e. $u(x) \in T_x S$ pour tout $x \in S$, où $TS = \cup_{x \in S} T_x S$ est le fibré tangent. Lorsque $\varepsilon \rightarrow 0$, ces champs tendent à être de module un (i.e. $|u|_g = 1$ sur S) et génèrent des points singuliers a_k appelés tourbillons. Les tourbillons a_k sont caractérisés par des indices (ou degrés topologiques) $d_k \in \mathbb{Z}$ qui quantifient l'énergie E_ε autour de a_k (i.e. $\pi d_k^2 |\log \varepsilon|$ au

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premier ordre) et satisfont la relation d'invariance topologique (3); de plus, les tourbillons a_k et leur degrés d_k sont détectés par la vortacité $\omega(u)$.

Notre premier objectif est de déterminer l'énergie d'interaction entre ces tourbillons (appelée énergie renormalisée) donnée par le développement à l'ordre deux de l'énergie E_ε . Ceci repose sur la notion de champ de vecteur canonique harmonique u^* (voir Section 2), qui dépend non seulement de la configuration $a = (a_k)$ et $d = (d_k)$, mais aussi d'un vecteur $\Phi \in \mathbb{R}^{2g}$ qui englobe les intégrales de flux de u^* (voir (8)). En effet, l'énergie renormalisée $W(a, d, \Phi)$ représente l'énergie de Dirichlet associée à u^* en dehors de petites boules centrées en a_k (voir le livre innovateur de Bethuel–Brezis–Hélein [3]). Nous calculons une formule exacte de $W(a, d, \Phi)$ (voir (14)) en utilisant les fonctions de Green en (a_k) ainsi que la fonction $\psi_0 = (-\Delta)^{-1}(-\kappa + \bar{\kappa})$, où κ est la courbure de Gauss sur S et $\bar{\kappa}$ est la moyenne de κ sur S . L'énergie renormalisée détermine la position optimale des tourbillons (a_k) pour les configurations limites u^* des minimiseurs u_ε de E_ε lorsque $\varepsilon \rightarrow 0$; dans le cas de la sphère unité S munie de la métrique standard, la formule (14) montre que les configurations optimales sont données par les couples de deux tourbillons (a_1, a_2) diamétralement opposés, de degrés $d_1 = d_2 = 1$.

Notre résultat principal consiste à établir la Γ -convergence de E_ε à l'ordre deux. Plus précisément, nous montrons la compacité des vorticités $\omega(u_\varepsilon)$ et des intégrales de flux $\Phi(u_\varepsilon)$ pour des champs de vecteurs u_ε d'énergie d'ordre $|\log \varepsilon|$. Ensuite, nous établissons les bornes inférieures et supérieures de E_ε à l'ordre deux qui font apparaître l'énergie renormalisée. Les preuves des résultats annoncés dans cette note font partie de notre article [9].

1. Introduction

Let (S, g) be a closed (i.e. compact, connected without boundary) oriented 2-dimensional Riemannian manifold of genus g . We will focus on (tangent) vector fields

$$u : S \rightarrow TS, \quad \text{i.e. } u(x) \in T_x S \text{ for every } x \in S$$

where $TS = \cup_{x \in S} T_x S$ is the tangent bundle of S . It is well known that there are no smooth vector fields $\mathcal{X}(S)$ (or more generally, of Sobolev regularity $\mathcal{X}^{1,2}(S)$) of unit length $|u|_g = 1$ on S (unless $g = 1$). In fact, vector fields of unit length have in general singular points with a (non-zero) index. Our aim is to determine the interaction energy between these singular points in a variational model of Ginzburg–Landau type depending on a small parameter $\varepsilon > 0$, where the penalty $|u|_g = 1$ in S is relaxed.

Model. For vector fields $u : S \rightarrow TS$, we define the energy functional

$$E_\varepsilon(u) = \int_S e_\varepsilon(u) \text{vol}_g, \quad e_\varepsilon(u) := \frac{1}{2} |Du|_g^2 + \frac{1}{4\varepsilon^2} F(|u|_g^2), \quad (1)$$

where $|Du|_g^2 := |D_{\tau_1} u|_g^2 + |D_{\tau_2} u|_g^2$ in S , vol_g is the volume 2-form on (S, g) and D_v denotes covariant differentiation (with respect to the Levi-Civita connection) of u in direction v and $\{\tau_1, \tau_2\}$ is any local orthonormal basis of TS . The potential $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function with $F(1) = 0$ and there exists some $c > 0$ such that $F(s^2) \geq c(1-s)^2$ for every $s \geq 0$; in particular, 1 is the unique zero of F . The parameter $\varepsilon > 0$ is small penalizing $|u|_g \neq 1$ in S ; the goal is to analyze the asymptotic behavior of E_ε in the framework of Γ -convergence (at first and second order) in the limit $\varepsilon \rightarrow 0$. This is a “toy” problem for some physical models arising for thin shells in micromagnetics and in nematic liquid crystals (see, e.g., [4,5]).

Connection 1-form. On an open subset $O \subset S$, a moving frame is a pair of smooth, properly oriented, orthonormal vector fields $\tau_k \in \mathcal{X}(O)$, $k = 1, 2$, i.e. $(\tau_k, \tau_l)_g = \delta_{kl}$, $k, l = 1, 2$, and $\text{vol}_g(\tau_1, \tau_2) = 1$ in O , where $(\cdot, \cdot)_g$ is the scalar product on TS . (We will use the same notation $(\cdot, \cdot)_g$ for the inner product associated with k -forms, $k = 0, 1, 2$.) Defining $i : TS \rightarrow TS$ such that i is an isometry of $T_x S$ to itself for every $x \in S$ satisfying

$$i^2 w = -w, \quad (iw, v)_g = -(w, iv)_g = \text{vol}_g(w, v),$$

then every smooth vector field $\tau \in \mathcal{X}(O)$ of unit length provides a moving frame $\{\tau_1, \tau_2\} := \{\tau, i\tau\}$ on O . Moreover, if $\{\tau_1, \tau_2\}$ is any moving frame in O , then $\tau_2 = i\tau_1$.¹ Given a moving frame $\{\tau_1, \tau_2\}$ on an open subset $O \subset S$, the *connection 1-form* A associated with $\{\tau_1, \tau_2\}$ is defined for every smooth vector field $v \in \mathcal{X}(O)$:

$$A(v) := (D_v \tau_2, \tau_1)_g = -(D_v \tau_1, \tau_2)_g \quad \text{in } O.$$

In particular, $D_v \tau_1 = -A(v)\tau_2$ and $D_v \tau_2 = A(v)\tau_1$ in O . The definition of A depends on the choice of the moving frame. However, the exterior derivative dA of the connection 1-form is independent of the moving frame; in particular, the following identity holds $dA = \kappa \text{vol}_g$, where κ is the Gaussian curvature of S (see [7], Proposition 2, Chapter 5.3). We recall the Gauss–Bonnet theorem, which states

¹ In general a moving frame exists only locally on S .

$$\int_S \kappa \operatorname{vol}_g = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic, related to the genus g of S by $\chi(S) = 2 - 2g$.

Vortices. We will identify vortices of a vector field u with small geodesic balls centered at some points around which u has a (non-zero) index. To be more precise, we introduce the Sobolev space $\mathcal{X}^{1,p}(S)$ of vector fields $u : S \rightarrow TS$ such that $|u|_g$ and $|Du|_g$ belong to $L^p(S)$ (with respect to the volume 2-form), $p \geq 1$. Given $u \in \mathcal{X}^{1,p}(S) \cap L^q(S)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$, we define the 1-form $j(u)$ by²

$$j(u) = (Du, iu)_g.$$

In particular, $j(u)$ is a well-defined 1-form in $L^1(S)$ if $u \in \mathcal{X}^{1,1}(S)$ with $|u|_g = 1$ almost everywhere in S ; the same is true if $u \in \mathcal{X}^{1,p}(S)$ for $p \geq \frac{4}{3}$. To introduce the notion of index, we assume that O is a simply connected open subset of S and $u \in \mathcal{X}^{1,2}(N)$ is a vector field in a neighborhood N of ∂O such that $|u|_g \geq \frac{1}{2}$ a.e. in N ; then the *index* (or winding number) of u along ∂O is defined by

$$\operatorname{deg}(u; \partial O) := \frac{1}{2\pi} \left(\int_{\partial O} \frac{j(u)}{|u|_g^2} + \int_O \kappa \operatorname{vol}_g \right)$$

(see [7], Chapter 6.1). In particular, if u is defined in $O \cup N$ and has unit length on ∂O , then one has $\int_O \omega(u) = 2\pi \operatorname{deg}(u; \partial O)$

where $\omega(u)$ is the *vorticity* associated with the vector field u :

$$\omega(u) := dj(u) + \kappa \operatorname{vol}_g. \tag{2}$$

Sometimes we can identify the index of u at a point $P \in S$ with the index of u along a curve around P . Note that every smooth vector field $u \in \mathcal{X}(O)$ (or more generally, $u \in \mathcal{X}^{1,2}(O)$) of unit length in O has $\operatorname{deg}(u; \partial O) = 0$; moreover, a vortex with non-zero index will carry infinite energy E_ε as $\varepsilon \rightarrow 0$.

We will prove a Γ -convergence result (at the second order) of E_ε as $\varepsilon \rightarrow 0$. In particular, at the level of minimizers u_ε of E_ε , we show that u_ε converges in $\mathcal{X}^{1,1}(S)$ (for a subsequence) to a canonical harmonic vector field u^* of unit length that is smooth³ away from $n = |\chi(S)|$ distinct singular points a_1, \dots, a_n , each singular point a_k carrying the same index $d_k = \operatorname{sign} \chi(S) \in \{\pm 1\}$ so that⁴

$$\sum_{k=1}^n d_k = \chi(S). \tag{3}$$

The vorticity $\omega(u^*)$ detects the singular points $\{a_k\}_{k=1}^n$ of u^* :

$$\omega(u^*) = 2\pi \sum_{k=1}^n d_k \delta_{a_k} \quad \text{in } S, \tag{4}$$

where δ_{a_k} is the Dirac measure (as a 2-form) at a_k . The expansion of the minimal energy E_ε at the second order is given by

$$E_\varepsilon(u_\varepsilon) = n\pi \log \frac{1}{\varepsilon} + \lim_{r \rightarrow 0} \left(\int_{S \setminus \cup_{k=1}^n B_r(a_k)} \frac{1}{2} |Du^*|_g^2 \operatorname{vol}_g + n\pi \log r \right) + n\gamma_F + o(1), \text{ as } \varepsilon \rightarrow 0,$$

where $\gamma_F > 0$ is a constant depending only on the potential F and $B_r(a_k)$ is the geodesic ball centered at a_k of radius r . The second term in the above RHS is called the *renormalized energy* between the vortices a_1, \dots, a_n and governs the optimal location of these singular points as in the Euclidean case (see the seminal book of Bethuel–Brézis–Hélein [3], which is essential for our note). In particular, if S is the unit sphere in \mathbb{R}^3 endowed with the standard metric g , then $n = 2$ and a_1 and a_2 are two diametrically opposed points on S .

Outline of the note. The note is divided as follows. Section 2 is devoted to characterize canonical harmonic vector fields of unit length. In Section 3, we determine the renormalized energy between singular points of canonical harmonic vector fields. The main Γ -convergence result is stated in the last section. The proofs of these results are part of our forthcoming article [9].

² Note that if $\{\tau_1, \tau_2\}$ is a moving frame on an open set $O \subset S$, then the connection 1-form A associated with the moving frame is given by $A = -j(\tau_1)$ on O . In particular, $dj(u) = -k \operatorname{vol}_g$ in O for every smooth $u \in \mathcal{X}(O)$ of unit length.

³ In the case of a surface (S, g) with genus 1 (i.e. homeomorphic with the flat torus), then $n = 0$ and u^* is smooth in S .

⁴ In fact, $\operatorname{deg}(u^*; \gamma) = d_k$ for every closed simple curve γ around a_k and lying near a_k .

2. Canonical harmonic vector fields of unit length

We will say that a canonical harmonic vector field of unit length having the singular points $a_1, \dots, a_n \in S$ of index $d_1, \dots, d_n \in \mathbb{Z}$ for some $n \geq 1$ is a vector field $u^* \in \mathcal{X}^{1,1}(S)$ such that $|u^*|_g = 1$ in S , (4) holds and

$$d^*j(u^*) = 0 \quad \text{in } S. \tag{5}$$

Here, d^* is the adjoint of the exterior derivative d , i.e. $d^*j(u^*)$ is the unique 0-form on S such that

$$\int_S (d^*j(u^*), \zeta)_g \text{vol}_g = \int_S (j(u^*), d\zeta)_g \text{vol}_g \quad \text{for every smooth 0-form } \zeta.$$

If u^* satisfies (4), then the Gauss–Bonnet theorem combined with (2) implies that necessarily (3) holds.

We will see that condition (3) is also sufficient. Indeed, if (3) holds, we will construct solutions to (4) and (5), as follows: let $\psi = \psi(a, d)$ be the unique 2-form on S solving:

$$-\Delta\psi = -\kappa \text{vol}_g + 2\pi \sum_{k=1}^n d_k \delta_{a_k} \quad \text{in } S, \quad \int_S \psi = 0, \tag{6}$$

with the sign convention that $-\Delta = dd^* + d^*d$. The idea is to find u^* such that $j(u^*) - d^*\psi$ is an harmonic 1-form, i.e. $Harm^1(S) = \{\text{integrable 1-forms } \eta \text{ on } S : d\eta = d^*\eta = 0 \text{ as distributions}\}$. The dimension of the space $Harm^1(S)$ is twice the genus (i.e. $2g$) of (S, g) and we fix an orthonormal basis η_1, \dots, η_{2g} of $Harm^1(S)$ such that $\int_S (\eta_k, \eta_l)_g \text{vol}_g = \delta_{kl}$ for $k, l = 1, \dots, 2g$. Therefore, it is expected that

$$j(u^*) = d^*\psi + \sum_{k=1}^{2g} \Phi_k \eta_k \quad \text{in } S \tag{7}$$

for some constant vector $\Phi = (\Phi_1, \dots, \Phi_{2g}) \in \mathbb{R}^{2g}$. These constants are called *flux integrals* as they can be recovered by

$$\Phi_k = \int_S (j(u^*), \eta_k)_g \text{vol}_g, \quad \text{for } k = 1, \dots, 2g. \tag{8}$$

Note that (7) combined with (6) automatically yields (4) and (5). One important point is to characterize for which values of Φ the RHS of (7) arises as $j(u^*)$ for some vector field u^* of unit length in S . For that condition, we need to recall the following classical fact (see for example [8]): there exist $2g$ simple closed geodesics γ_ℓ on S , $\ell = 1, \dots, 2g$, such that for any closed Lipschitz curve γ on S , one can find integers c_1, \dots, c_{2g} such that γ is homologous to $\sum_{\ell=1}^{2g} c_\ell \gamma_\ell$, i.e. there exists an

integrable function $f : S \rightarrow \mathbb{Z}$ such that $\int_\gamma \zeta - \sum_{\ell=1}^{2g} c_\ell \int_{\gamma_\ell} \zeta = \int_S f d\zeta$ for all smooth 1-forms ζ . Having chosen the geodesic curves $\{\gamma_\ell\}_{\ell=1}^{2g}$ and the harmonic 1-forms $\{\eta_k\}_{k=1}^{2g}$, we fix the notation

$$\alpha_{\ell k} := \int_{\gamma_\ell} \eta_k, \quad k, \ell = 1, \dots, 2g. \tag{9}$$

Theorem 1. *Let $n \geq 1$ and $d = (d_1, \dots, d_n) \in \mathbb{Z}^n$ satisfy (3). Then for every $a = (a_1, \dots, a_n) \in S^n$, there exists*

$$\zeta_\ell = \zeta_\ell(a; d) \in \mathbb{R}/2\pi\mathbb{Z}, \quad \ell = 1, \dots, 2g$$

such that if a vector field $u^* \in \mathcal{X}^{1,1}(S)$ of unit length solves (4) and (5), then $j(u^*)$ has the form (7) for constants Φ_1, \dots, Φ_{2g} such that

$$\sum_{k=1}^{2g} \alpha_{\ell k} \Phi_k + \zeta_\ell(a, d) \in 2\pi\mathbb{Z}, \quad \ell = 1, \dots, 2g, \tag{10}$$

where $(\alpha_{\ell k})$ were defined in (9). Conversely, given any Φ_1, \dots, Φ_{2g} satisfying (10), there exists a vector field $u^* \in \mathcal{X}^{1,1}(S)$ of unit length solving (4) and (5) and such that $j(u^*)$ satisfies (7). In addition, the following hold:

1) $\zeta_\ell(\cdot; d)$ depends continuously on $a \in S^n$ for every $\ell = 1, \dots, 2g$. More generally, if⁵

$$\mu^t := 2\pi \sum_{l=1}^{n_t} d_l^t \delta_{a_l^t} \rightarrow \mu^0 := 2\pi \sum_{l=1}^{n_0} d_l^0 \delta_{a_l^0} \quad \text{in } W^{-1,1} \quad \text{as } t \downarrow 0,$$

$\{d_l^t\}_l$ are integers with (3) and $\sum_{l=1}^{n_t} |d_l^t|$ is uniformly bounded in t , then $\zeta_\ell(a^t, d^t) \rightarrow \zeta_\ell(a^0, d^0)$ as $t \downarrow 0$.

2) Any u^* solving (4) and (5) belongs to $\mathcal{X}^{1,p}(S)$ for all $1 \leq p < 2$, and is smooth away from $\{a_k\}_{k=1}^n$.

3) If u^*, \tilde{u}^* both satisfy (7) for the same (a, d) and the same $\{\Phi_k\}_{k=1}^{2g}$, then $\tilde{u}^* = e^{i\beta} u^*$ for some $\beta \in \mathbb{R}$.

The constants $\{\zeta_\ell(a; d)\}_{\ell=1}^{2g}$ are determined as follows. For every $\ell = 1, \dots, 2g$, we let λ_ℓ be some smooth simple closed curve such that λ_ℓ is homologous to γ_ℓ (the geodesics fixed in (9)) so that $\{a_k\}_{k=1}^n$ is disjoint from λ_ℓ ; for example, λ_ℓ is either γ_ℓ or, if γ_ℓ intersects some a_k , a small perturbation thereof. We now define $\zeta_\ell(a, d)$ to be the element of $\mathbb{R}/2\pi\mathbb{Z}$ such that

$$\zeta_\ell(a, d) := \int_{\lambda_\ell} (d^* \psi + A) \quad \text{mod } 2\pi, \quad \ell = 1, \dots, 2g, \tag{11}$$

where $\psi = \psi(a, d)$ is the 2-form given by (6) and A is the connection 1-form associated with any moving frame defined in a neighborhood of λ_ℓ . The integral in (11) is independent, modulo $2\pi\mathbb{Z}$, of the choice of moving frame and of the curve λ_ℓ homologous to γ_ℓ . In examples in which it can be explicitly computed, in general $\zeta_\ell(a, d) \neq 0 \pmod{2\pi}$ for $\ell = 1, \dots, 2g$.

3. Renormalized energy

For any $n \geq 1$, we consider n **distinct** points $a = (a_1, \dots, a_n) \in S^n$. Let $d = (d_1, \dots, d_n) \in \mathbb{Z}^n$ satisfying (3), $\{\zeta_\ell(a; d)\}_{\ell=1}^{2g}$ be given in Theorem 1 and $\Phi \in \mathbb{R}^{2g}$ be a constant vector belonging to the set:

$$\mathcal{L}(a, d) := \{ \Phi = (\Phi_1, \dots, \Phi_{2g}) \in \mathbb{R}^{2g} : \sum_{k=1}^{2g} \alpha_{\ell k} \Phi_k + \zeta_\ell(a, d) \in 2\pi\mathbb{Z}, \ell = 1, \dots, 2g \}.$$

We define the *renormalized energy* between the vortices a of indices d by

$$W(a, d, \Phi) := \lim_{r \rightarrow 0} \left(\int_{S \setminus \cup_{k=1}^n B_r(a_k)} \frac{1}{2} |Du^*|_g^2 \text{vol}_g + \pi \log r \sum_{k=1}^n d_k^2 \right), \tag{12}$$

where $u^* = u^*(a, d, \Phi)$ is the unique (up to a multiplicative complex number) canonical harmonic vector field given in Theorem 1 and $B_r(a_k)$ is the geodesic ball centered at a_k of radius r . Our arguments show that the above limit indeed exists. As in the Euclidean case (see [3]), we can compute the renormalized energy by using the Green’s function. For that, let $G(x, y)$ be the unique function on $S \times S$ such that for every $y \in S$:

$$-\Delta_x(G(\cdot, y) \text{vol}_g) = \delta_y - \frac{\text{vol}_g}{\text{Vol}_g(S)} \quad \text{distributionally in } S, \quad \int_S G(x, y) \text{vol}_g(x) = 0,$$

with $\text{Vol}_g(S) := \int_S \text{vol}_g$. Then G may be represented in the form (see [2], Chapter 4.2):

$$G(x, y) = G_0(x, y) + H(x, y), \quad \text{with } H \in C^1(S \times S),$$

where G_0 is smooth away from the diagonal, with

$$G_0(x, y) = -\frac{1}{2\pi} \log(\text{dist}(x, y)) \quad \text{if the geodesic distance } \text{dist}(x, y) < \frac{1}{2} \text{ (injectivity radius of } S).$$

The 2-form $\psi = \psi(a, d)$ defined in (6) can be written as:

$$\psi = 2\pi \sum_{k=1}^n d_k G(\cdot, a_k) \text{vol}_g + \psi_0 \text{vol}_g \quad \text{in } S,$$

⁵ If μ is a 2-form (possibly measure-valued) then we write for $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$\|\mu\|_{W^{-1,p}} := \sup \left\{ \int_S f \mu : f \in W^{1,q}(S; \mathbb{R}), \|f\|_{W^{1,q}} := \|f\|_{L^q} + \|df\|_{L^q} \leq 1 \right\}.$$

where $\psi_0 \in C^\infty(S)$ has zero average on S and solves

$$-\Delta\psi_0 = -\kappa + \bar{\kappa}, \quad \text{for } \bar{\kappa} = \frac{1}{\text{Vol}(S)} \int_S \kappa \text{vol}_g = \frac{2\pi\chi(S)}{\text{Vol}(S)}. \tag{13}$$

In other words, the 2-form $x \mapsto \psi(x) + d_k \log(\text{dist}(x, a_k) \text{vol}_g)$ is C^1 in a neighborhood of a_k for every $1 \leq k \leq n$. We have the following expression of the renormalized energy defined in (12).

Proposition 2. Given $n \geq 1$ distinct points $a_1, \dots, a_n \in S$, integers d_1, \dots, d_n with (3) and $\Phi \in \mathcal{L}(a, d)$, then

$$W(a, d, \Phi) = 4\pi^2 \sum_{l \neq k} d_l d_k G(a_l, a_k) + 2\pi \sum_{k=1}^n \left[\pi d_k^2 H(a_k, a_k) + d_k \psi_0(a_k) \right] + \frac{1}{2} |\Phi|^2 + \int_S \frac{|\text{d}\psi_0|^2}{2} \text{vol}_g, \tag{14}$$

where ψ_0 is defined in (13).

For the unit sphere S in \mathbb{R}^3 endowed with the standard metric (in particular, ψ_0 vanishes in S , $g = 0$ and $\Phi = 0$), if $n = 2$ and $d_1 = d_2 = 1$, then the second term in the RHS of (14) is independent of a_k (as $x \mapsto H(x, x)$ is constant, see [14]); thus, minimizing W is equivalent by minimizing the Green’s function $G(a_1, a_2)$ over the set of pairs (a_1, a_2) in $S \times S$, namely, the minimizing pairs are diametrically opposed.

4. Γ -convergence

Given the potential F in Section 1, we compute the energy E_ε of the radial profile of a vortex of index 1 inside a geodesic ball of radius $R > 0$:

$$I_F(R, \varepsilon) := \inf \left\{ \pi \int_0^R \left[f'(r)^2 + \frac{f(r)^2}{r^2} + \frac{1}{2\varepsilon^2} F(f(r)^2) \right] r \, dr : f(0) = 0, f(R) = 1 \right\}.$$

Then $I_F(R, \varepsilon) = I_F(\lambda R, \lambda \varepsilon) = I_F(1, \frac{\varepsilon}{R}) =: I_F(\frac{\varepsilon}{R})$ for every $\lambda > 0$, and the following limit exists (see [3]):

$$\gamma_F := \lim_{t \rightarrow 0} (I_F(t) + \pi \log t).$$

We state our main result as follows.

Theorem 3. The following Γ -convergence result holds.

- 1) (Compactness) Let $(u_\varepsilon)_{\varepsilon \downarrow 0}$ be a family of vector fields in $\mathcal{X}^{1,2}(S)$ satisfying $E_\varepsilon(u_\varepsilon) \leq N\pi |\log \varepsilon| + C$ for some integer $N \geq 0$ and a constant $C > 0$. We define

$$\Phi(u_\varepsilon) := \left(\int_S (j(u_\varepsilon), \eta_1)_g \text{vol}_g, \dots, \int_S (j(u_\varepsilon), \eta_{2g})_g \text{vol}_g \right) \in \mathbb{R}^{2g}.$$

Then there exists a sequence $\varepsilon \downarrow 0$ such that

$$\omega(u_\varepsilon) \longrightarrow 2\pi \sum_{k=1}^n d_k \delta_{a_k} \quad \text{in } W^{-1,1}, \text{ as } \varepsilon \rightarrow 0, \tag{15}$$

where $\{a_k\}_{k=1}^n$ are distinct points in S and $\{d_k\}_{k=1}^n$ are nonzero integers satisfying (3) and $\sum_{k=1}^n |d_k| \leq N$. Moreover, if $\sum_{k=1}^n |d_k| = N$, then $n = N$, $|d_k| = 1$ for every $k = 1, \dots, n$ (in particular, $n = \chi(S)$ modulo 2) and there exists $\Phi \in \mathcal{L}(a, d)$ such that $\Phi(u_\varepsilon) \rightarrow \Phi$ for a further sequence $\varepsilon \rightarrow 0$.

- 2) (Γ -liminf) Assume that the vector fields $u_\varepsilon \in \mathcal{X}^{1,2}(S)$ satisfy (15) for n distinct points $\{a_k\}_{k=1}^n \in S^n$ and $|d_k| = 1, k = 1, \dots, n$ that satisfy (3) and $\Phi(u_\varepsilon) \rightarrow \Phi$ for some $\Phi \in \mathcal{L}(a, d)$. Then

$$\liminf_{\varepsilon \rightarrow 0} [E_\varepsilon(u_\varepsilon) - n\pi |\log \varepsilon|] \geq W(a, d, \Phi) + n\gamma_F.$$

- 3) (Γ -limsup) For every n distinct points $a_1, \dots, a_n \in S$ and $d_1, \dots, d_n \in \{\pm 1\}$ satisfying (3) and every $\Phi \in \mathcal{L}(a, d)$, there exists a sequence of vector fields u_ε on S such that (15) holds, $\Phi(u_\varepsilon) \rightarrow \Phi$ and

$$E_\varepsilon(u_\varepsilon) - n\pi |\log \varepsilon| \longrightarrow W(a, d, \Phi) + n\gamma_F \quad \text{as } \varepsilon \rightarrow 0.$$

This theorem is the generalization of the Γ -convergence result for E_ε in the Euclidean case (see [6,11,13,1]) and it is based on topological methods for energy concentration (vortex ball construction, vorticity estimates etc.) as introduced in [10,12].

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